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ANALYTICAL SOLUTION OF THE TIME-DEPENDENT KINETIC EQUATION FOR A DIATOMIC GAS†

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An analytical solution of the time-dependent kinetic equation for a diatomic gas is obtained. The problem of a point source of heat or particles is considered as an application. © 2005 Elsevier Ltd. All rights reserved.

For problems related to transfer phenomena in molecular gases [1], analytical solutions have only been obtained at present in the time-independent case [2–6].

Consider the equation

$$\frac{\partial\varphi}{\partial t} + C_x\frac{\partial\varphi}{\partial x} = I[\varphi] \tag{1}$$

Here φ is the correction to the equilibrium (Maxwell) distribution function, which in the case of a diatomic gas considered here, is given by the relation

$$f_0 = n_0\left(\frac{m}{2\pi kT_0}\right)^{3/2}\frac{J}{kT_0}\exp(-C^2-\gamma^2), \quad C = V\sqrt{\frac{m}{2kT_0}}, \quad \gamma = \omega\sqrt{\frac{J}{2kT_0}}$$

where V and ω are the natural velocity of translational and rotational motion of the gas molecules, m and J are the mass and moment of inertia of the molecules, k is Boltzmann's constant, and T_0 and n_0 are the unperturbed values of the temperature and density.

We will assume [2]

$$I[\varphi] = \sum_{i=1}^{3} P_iM_i - \varphi$$

where

$$M_i = 2\pi^{-3/2}\int P_i\varphi\exp(-C^2-\gamma^2)\gamma d\gamma d^3C$$

$$P_1 = 1, \quad P_2 = \sqrt{\frac{2}{5}}\left(C^2+\gamma^2-\frac{5}{2}\right), \quad P_3 = \sqrt{2}C_x$$

We will put $C_x = \mu$ and represent φ in the form

$$\begin{aligned}\varphi &= e_1Y_1(t, x, \mu) + e_2Y_2(t, x, \mu)\\ e_1 &= 1, \quad e_2 = \frac{1}{\nu}(C^2-\mu^2+\gamma^2-\nu^2), \quad \nu = \sqrt{2}\end{aligned} \tag{2}$$

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As a result, Eq. (1) is reduced to an integro-differential equation in the vector $\mathbf{Y} = \left\| \begin{matrix} Y_1 \\ Y_2 \end{matrix} \right\|$:

$$\left(\frac{\partial}{\partial t} + \mu\frac{\partial}{\partial x} + 1\right)\mathbf{Y}(t, x, \mu) = \pi^{-1/2}\int_{-\infty}^{+\infty} \mathbf{K}(\mu, \mu_1)\mathbf{Y}(t, x, \mu_1)\exp(-\mu_1^2)d\mu_1$$

$$\mathbf{K}(\mu, \mu_1) = \left\| \begin{matrix} 1 + 2\mu\mu_1 + \frac{2}{5}\left(\mu^2 - \frac{1}{2}\right)\left(\mu_1^2 - \frac{1}{2}\right) & \frac{2}{5}\nu\left(\mu^2 - \frac{1}{2}\right) \\ \frac{2}{5}\nu\left(\mu_1^2 - \frac{1}{2}\right) & \frac{2}{5}\nu^2 \end{matrix} \right\|$$

Separating the variables, as in the well-known approach [7], we can represent the solution of this equation in the form

$$\mathbf{Y}(t, x, \mu) = \exp(\sigma t - (\sigma + 1)x/\eta)\mathbf{F}(\sigma, \eta, \mu)$$

The components of the vector $\mathbf{F}$ are found from the system of characteristic equations

$$\begin{aligned} \pi^{1/2}\left(1 - \frac{\mu}{\eta}\right)(\sigma + 1)F_1 &= N_1^0 + 2\mu N_1^1 + \frac{2\mu^2 - 1}{10}(2N_1^2 - N_1^0 + 2\nu N_2^0) \\ \pi^{1/2}\left(1 - \frac{\mu}{\eta}\right)(\sigma + 1)F_2 &= \frac{\nu}{5}(2N_1^2 - N_1^0 + 2\nu N_2^0) \end{aligned} \tag{3}$$

$$N_i^\alpha = \int_{-\infty}^{+\infty} F_i\mu^\alpha\exp(-\mu^2)d\mu \tag{4}$$

Following the procedure proposed earlier in [8], we express the higher moments of the function F_1 occurring in system (3), (4) in terms of N_1^0. To do this we multiply the first equation of (3) by $\exp(-\mu^2)$ and $\mu\exp(-\mu^2)$ and integrate over the whole range of variation μ. Solving the system of equations obtained, we find

$$N_1^\alpha = N_1^0\left(\frac{\eta\sigma}{\sigma + 1}\right)^\alpha$$

Hence, Eqs (3) and (4) can also be represented in the vector form

$$\pi^{1/2}(\eta - \mu)(\sigma + 1)\mathbf{F} = \eta\mathbf{\Delta}\mathbf{N} \tag{5}$$

$$\mathbf{\Delta} = \mathbf{K}\left(\mu, \frac{\eta\sigma}{\sigma + 1}\right) \tag{6}$$

$$\mathbf{N} = \left\| \begin{matrix} N_1^0 \\ N_2^0 \end{matrix} \right\| = \int_{-\infty}^{+\infty} \mathbf{F}\exp(-\mu^2)d\mu \tag{7}$$

Equation (7) can be regarded as the normalization condition for the function $\mathbf{F}$.

When η is not a real number, we obtain from Eq. (5)

$$\mathbf{F} = \frac{\pi^{-1/2}\eta}{(\eta - \mu)(\sigma + 1)}\mathbf{\Delta}\mathbf{N} \tag{8}$$

The values of η corresponding to this solution are defined by condition (7), which gives

$$\mathbf{N} = \frac{\pi^{-1/2}\eta}{\sigma+1}\int_{-\infty}^{+\infty}\Delta\exp(-\mu^2)\frac{d\mu}{\eta-\mu}\mathbf{N}$$

A non-trivial solution of this equation exists when

$$\det\mathbf{\Lambda}(\sigma,\eta) = 0 \tag{9}$$

Here

$$\Lambda(\sigma,\eta) = (\sigma+1)\mathbf{E} - \pi^{-1/2}\eta\int_{-\infty}^{+\infty}\Delta\exp(-\mu^2)\frac{\mu}{\eta-\mu} =$$

$$= \begin{Vmatrix} \sigma+1+\lambda_c(\eta)+\frac{2\eta^2\sigma}{\sigma+1}(\lambda_c(\eta)+1)+\lambda_1 l & \nu\lambda_1 \\ \frac{2}{5}\nu\lambda_c(\eta)l & \sigma+1+\frac{4}{5}\lambda_c(\eta) \end{Vmatrix} \tag{10}$$

$$\lambda_1 = \frac{\lambda_c(\eta)(2\eta^2-1)+2\eta^2}{5}, \quad l = \left(\frac{\eta\sigma}{\sigma+1}\right)^2-\frac{1}{2}$$

$$\lambda_c(z) = \pi^{-1/2}z\int_{-\infty}^{+\infty}\exp(-\mu^2)\frac{d\mu}{\mu-z}$$

where **E** is the identity matrix. Then, the normalized vector **N** itself can be defined, apart from an arbitrary constant, by the equation

$$\mathbf{N} = \mathrm{const}\begin{Vmatrix} \Lambda_{22} \\ -\Lambda_{21} \end{Vmatrix} \tag{11}$$

To solve the dispersion equation (9) we make use of the theory of boundary-value problems of the function of a complex variable (see, for example, [9]). Note that $D(z) = \det\Lambda(\sigma, z)$ is an even piecewise-analytical function in the complex plane with a cut along the real axis. We will denote the contractions of this function in the upper and lower half-planes by D^+ and D^- respectively. Consider the homogeneous Riemann boundary-value problem

$$X^+(x) = G(x)X^-(x), \quad x \in \mathbb{R} \tag{12}$$

with the coefficient $G(x) = D^+(x)/D^-(x)$, where $D^{\pm}(x) = \lim D(x+iy)$ when $y \to \pm 0$.

By virtue of the generalized Liouville theorem, the general solution of problem (12) is given by the expression

$$D^{\pm}(z) = A(z+i)^{-2\kappa}X^{\pm}(z)\prod_{\alpha=1}^{\kappa}(\eta_\alpha^2-z^2)$$

where

$$X^{\pm}(z) = \left(\frac{z+i}{z\pm i}\right)^{2\kappa}\exp(\Gamma(z))$$

$$\Gamma(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\ln\left(\left(\frac{\mu+i}{\mu-i}\right)^{2\kappa}G(\mu)\right)\frac{d\mu}{\mu-z}$$

$$\kappa = \frac{1}{2}\mathrm{ind}G(x) = \frac{1}{2\pi}[\arg G(x)]_{\mathbb{R}_+}, \quad \mathbb{R}_+ \in (0, \infty)$$

Hence, to determine η_α it is sufficient to calculate $D^\pm$ and $X^\pm$ for arbitrary $\kappa + 1$ values of z and solve the system obtained for these quantities and the constant A. The values of $\pm\eta_\alpha$ obtained are the required roots of Eq. (9).

Taking expression (11) into account, we obtain from relation (8)

$$\mathbf{F}(\sigma, \eta_\alpha, \mu) = \frac{\pi^{-1/2}\eta_\alpha}{(\eta_\alpha - \mu)(\sigma + 1)}\left\| \begin{matrix} \Lambda_{22}\Delta_{11} - \Lambda_{21}\Delta_{12} \\ \Lambda_{22}\Delta_{21} - \Lambda_{21}\Delta_{22} \end{matrix} \right\| \tag{13}$$

where Λ_{ij} and Δ_{ij} are the components of the matrices (10) and (6), calculated for $\eta = \eta_\alpha$.

In the case of real values of η, the following functions are solutions of Eqs (5) and (7)

$$\mathbf{\Phi}(\sigma, \eta, \mu) = \left(\pi^{-1/2}\frac{\eta}{\eta - \mu}\mathbf{\Delta} + \exp(\eta^2)\mathbf{\Lambda}\delta(\eta - \mu)\right)\frac{\mathbf{N}}{\sigma + 1} \tag{14}$$

where all the integrals of the function (14) must be evaluated in the sense of the principal value of the Cauchy integral.

In view of the arbitrary nature of the normalized vector **N**, the solution (14) can be represented as the superposition of two independent functions

$$\mathbf{\Phi}_1 = \pi^{-1/2}\frac{\eta}{\eta - \mu}\left\| \begin{matrix} \mu^2 - \frac{1}{2} \\ \nu \end{matrix} \right\| + \exp(\eta^2)\delta(\eta - \mu)\left\| \begin{matrix} \lambda_p(\eta)\left(\eta^2 - \frac{1}{2}\right) + \eta^2 \\ \frac{5}{2\nu}(\sigma + 1) + \nu\lambda(\eta) \end{matrix} \right\|$$

$$\mathbf{\Phi}_2 = \pi^{-1/2}\frac{\eta}{\eta - \mu}\left\| \begin{matrix} 1 + \frac{2\mu\eta\sigma}{\sigma + 1} \\ 0 \end{matrix} \right\| + \exp(\eta^2)\delta(\eta - \mu)\left\| \begin{matrix} \sigma + 1 + \lambda_p(\eta) + 2(\lambda_p(\eta) + 1)\frac{\eta^2\sigma}{\sigma + 1} \\ \frac{\sigma + 1}{2\nu} - \frac{\sigma^2\eta^2}{(\sigma + 1)\nu} \end{matrix} \right\|$$

Here

$$\lambda_p(\eta) = -2\eta\exp(-\eta^2)\int_0^\eta \exp(\mu^2)d\mu$$

The functions $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ constitute the continuous spectrum of solutions of Eq. (5).

It can be proved (see, for example, [7]), that the system of equations obtained represents a complete system of orthogonal functions, which satisfy the conditions

$$\int_{-\infty}^{+\infty} \mathbf{F}(\sigma, \eta_\alpha, \mu)\mathbf{F}(\sigma, \eta_\beta, \mu)\exp(-\mu^2)\mu d\mu = \delta_{\alpha\beta}N_\alpha$$

$$\int_{-\infty}^{+\infty} \mathbf{F}(\sigma, \eta_\alpha, \mu)\mathbf{\Phi}_\beta(\sigma, \eta, \mu)\exp(-\mu^2)\mu d\mu = 0 \tag{15}$$

$$\int_{-\infty}^{+\infty} \mathbf{X}_\alpha(\sigma, \eta', \mu)\mathbf{\Phi}_\beta(\sigma, \eta, \mu)\exp(-\mu^2)\mu d\mu = \delta_{\alpha\beta}\delta(\eta - \eta')N_0$$

where

$$\mathbf{X}_1 = N_{11}\mathbf{\Phi}_1 - N_{12}\mathbf{\Phi}_2, \quad \mathbf{X}_2 = N_{22}\mathbf{\Phi}_2 - N_{12}\mathbf{\Phi}_1$$

$$N_{11} = \left(\sigma + 1 + \lambda_p(\eta) + 2(\lambda_p(\eta) + 1)\frac{\eta^2\sigma}{\sigma + 1}\right)^2 +$$

$$+\frac{1}{2}\left(\frac{\sigma + 1}{2} - \frac{\eta^2\sigma^2}{\sigma + 1}\right)^2 + \pi\eta^2\exp(-2\eta^2)\left(1 + \frac{2\eta^2\sigma}{\sigma + 1}\right)$$

$$N_{12} = \left(\frac{\sigma + 1}{2} - \frac{\eta^2\sigma^2}{\sigma + 1}\right)\left(\frac{5}{4}(\sigma + 1) + \lambda_p(\eta)\right) +$$

$$+\left(\sigma + 1 + \lambda_p(\eta) + 2(\lambda_p(\eta) + 1)\frac{\eta^2\sigma}{\sigma + 1}\right)\left(\lambda_p(\eta)\left(\eta^2 - \frac{1}{2}\right) + \eta^2\right) +$$

$$+\pi\eta^2\exp(-2\eta^2)\left(\eta^2 - \frac{1}{2}\right)\left(1 + \frac{2\eta^2\sigma}{\sigma + 1}\right)$$

$$N_{22} = \left(\lambda_p(\eta)\left(\eta^2 - \frac{1}{2}\right) + \eta^2\right)^2 + 2\left(\frac{5}{4}(\sigma + 1) + \lambda_p(\eta)\right)^2 +$$

$$+\pi\eta^2\exp(-2\eta^2)\left(\left(\eta^2 - \frac{1}{2}\right)^2 + 2\right)$$

$$N_0 = \eta\exp(\eta^2)(N_{11}N_{22} - N_{12}^2)$$

In view of the extremely long expressions for N_α it is best to calculate them directly by numerical integration.

We will consider, as an application of the solution obtained, an infinite plane heat source of power $W(t) = \exp(\sigma t)$, situated in the $x = 0$ plane.

The distribution function in this case can be represented in the form

$$\mathbf{Y}(t, x, \mu) = \mathbf{Y}^{\pm}(\sigma, x, \mu)\exp(\sigma t) \quad \text{for} \quad \pm x > 0$$

where

$$\mathbf{Y}^{\pm}(\sigma, x, \mu) = \pm\sum_{\alpha} A_\alpha^{\pm}\mathbf{F}_\alpha(\sigma, \pm\eta_\alpha, \mu)\exp(\mp(\sigma + 1)x/\eta_\alpha) + \\ + \sum_{\beta = 0}^{2} \int_0^{\pm\infty} B_\beta\mathbf{\Phi}_\beta(\sigma, \eta, \mu)\exp(-(\sigma + 1)x/\eta)d\eta \tag{16}$$

where the summation in the first term must be carried out only over those values of α for which $\mathrm{Re}((\sigma + 1)/\eta_\alpha) > 0$.

The coefficients A and B are found from the jump condition

$$\mu(\mathbf{Y}^+ - \mathbf{Y}^-) = \mathbf{S} \quad \text{when} \quad x = 0$$

Hence, by virtue of conditions (15), we obtain

$$A_\alpha^{\pm} = \int_{-\infty}^{+\infty} \mathbf{F}(\sigma, \pm\eta_\alpha, \mu)\mathbf{S}\exp(-\mu^2)d\mu, \quad B_\beta = \int_{-\infty}^{+\infty} \mathbf{X}_\beta\mathbf{S}\exp(-\mu^2)d\mu \tag{17}$$

In the case considered

$$\mathbf{S} = \mathbf{S}_h = \frac{2}{5}\begin{Vmatrix} \mu^2 - \frac{1}{2} \\ \nu \end{Vmatrix}$$

Correspondingly

$$\begin{aligned}
&A_\alpha^\pm = \frac{a_\alpha^h}{N_\alpha}, \quad B_1 = \frac{\sigma+1}{N_0}(N_{11}b_1^h - N_{12}b_2^h), \quad B_2 = \frac{\sigma+1}{N_0}(N_{22}b_2^h - N_{12}b_1^h) \\
&a_\alpha^h = \frac{2\eta_\alpha^2\sigma^2}{5\sigma+1} - \frac{\sigma+1}{5}, \quad b_1^h = 1, \quad b_2^h = 0 \\
&\mathbf{Y}_h^\pm(\sigma, x, \mu) = \pm\sum_\alpha \frac{a_\alpha^h}{N_\alpha}\mathbf{F}_\alpha(\sigma, \pm\eta_\alpha, \mu)\exp(\mp(\sigma+1)x/\eta_\alpha) + \\
&+ \int_0^{\pm\infty} \frac{\sigma+1}{N_0}\mathbf{X}_1 \exp(-(\sigma+1)x/\eta)d\eta
\end{aligned} \tag{18}$$

For relative temperature drops

$$\Delta T = \frac{T - T_0}{T_0} = \frac{4}{5}\pi^{-3/2}\int\left(C^2 + \gamma^2 - \frac{5}{2}\right)\varphi\exp(-C^2 - \gamma^2)\gamma d\gamma d^3C$$

and a density of the gas molecules

$$\Delta N = \frac{n - n_0}{n_0} = 2\pi^{-3/2}\int\varphi\exp(-C^2 - \gamma^2)\gamma d\gamma d^3C$$

we have

$$\Delta T_h = \pi^{-1/2}\exp(\sigma t)\left(\pm\sum_\alpha \frac{(a_\alpha^h)^2}{N_\alpha}\exp(\mp(\sigma+1)x/\eta_\alpha) + \right.$$
$$\left. + (\sigma+1)^2\int_0^{\pm\infty} \frac{N_{11}}{N_0}\exp(-(\sigma+1)x/\eta)d\eta\right)$$

$$\Delta N_h = \pi^{-1/2}\exp(\sigma t)\left(\pm\sum_\alpha \frac{a_\alpha^h a_\alpha^p}{N_\alpha}\exp(\mp(\sigma+1)x/\eta_\alpha) - \right.$$
$$\left. - (\sigma+1)^2\int_0^{\pm\infty} \frac{N_{12}}{N_0}\exp(-(\sigma+1)x/\eta)d\eta\right)$$

In the case of separate excitation of the translational and rotational degrees of freedom

$$S_\upsilon = \frac{2}{3}C^2 - 1 \quad \text{and} \quad S_\omega = \gamma^2 - 1$$

the vectors $\mathbf{F}$ and $\mathbf{X}_\beta$ must be represented in the expanded form

$$\mathbf{F} = e_1F_1 + e_2F_2, \quad \mathbf{X}_\beta = e_1X_{1\beta} + e_2X_{2\beta}$$

The following relations are then satisfied

$$A_\alpha^{\pm} = \frac{2}{\pi}\int (e_1 F_1(\sigma, \pm\eta_\alpha, \mu) + e_2 F_2(\sigma, \pm\eta_\alpha, \mu)) S \exp(-C^2 - \gamma^2)\gamma d\gamma d^3 C$$

$$B_\beta = \frac{2}{\pi}\int (e_1 X_{1\beta} + e_2 X_{2\beta}) S \exp(-C^2 - \gamma^2)\gamma d\gamma d^3 C$$

It is of some interest to calculate the relative temperature drop corresponding to the average energy of translational and rotational motion of the molecules:

$$\Delta T^{\upsilon} = \frac{4}{3}\pi^{-3/2}\int\left(C^2 - \frac{3}{2}\right)\varphi \exp(-C^2 - \gamma^2)\gamma d\gamma d^3 C$$

$$\Delta T^{\omega} = 2\pi^{-3/2}\int(\gamma^2 - 1)\varphi \exp(-C^2 - \gamma^2)\gamma d\gamma d^3 C$$

To complete the picture we must consider

$$\mathbf{S}_p = \left\| \begin{matrix} 1 \\ 0 \end{matrix} \right\|$$

which corresponds to a source of particles.

The values of the macroscopic parameters of the gas in these cases are given by the relation

$$M_s^m(t, \sigma, x) = \pm\pi^{-1/2}\sum_\alpha \frac{a_\alpha^s a_\alpha^m}{N_\alpha}\exp(\sigma t \mp (\sigma + 1)x/\eta_\alpha) +$$

$$+ \pi^{-1/2}(\sigma + 1)^2 \int_0^{\pm\infty} \frac{N_{11} b_1^s b_1^m + N_{22} b_2^s b_2^m - N_{12}(b_1^s b_2^m + b_2^s b_1^m)}{N_0}\exp(\sigma t - (\sigma + 1)x/\eta) d\eta \qquad (19)$$

Here and henceforth the superscript s indicates the nature of the source: $s = 1 - S_h$, $2 - S_\upsilon$, $3 - S_\omega$, $4 - S_p$, and the values of $m = 1, 2, 3, 4$ correspond to ΔT, ΔT^υ, ΔT^ω, ΔN. We then have

$$b_1^2 = \frac{5}{6}, \quad b_1^3 = \frac{5}{4}, \quad b_1^4 = 0$$

$$b_2^2 = \frac{1}{3}\left(\frac{\eta\sigma}{\sigma + 1}\right)^2 - \frac{1}{6}, \quad b_2^3 = \frac{1}{4} - \frac{1}{2}\left(\frac{\eta\sigma}{\sigma + 1}\right)^2, \quad b_2^4 = 1$$

$$a_\alpha^2 = \frac{10(\sigma + 1) + \lambda_c(\eta_\alpha)}{30}\left(2\left(\frac{\eta_\alpha\sigma}{\sigma + 1}\right)^2 - 1\right)$$

$$a_\alpha^3 = \frac{\lambda_c(\eta_a)}{20}\left(1 - 2\left(\frac{\eta_\alpha\sigma}{\sigma + 1}\right)^2\right), \quad a_\alpha^4 = \sigma + 1 + \frac{4}{5}\lambda_c(\eta_\alpha)$$

The coefficients a_α^1, b_1^1 and b_2^1 are given by relations (18). In passing we must draw attention to the symmetry of the moments of the distribution function with respect to an interchange of the upper and lower subscripts, i.e. $M_s^m = M_m^s$.

It is obvious that a plane source can be considered as a system of isotropic point sources. Consequently, in the (linear) approximation considered, the distribution of any scalar quantity ρ_{pl} can be expressed in terms of the distribution of this quantity, produced by a point source ρ_{pl}, i.e.

$$\rho_{pl}(x) = \int \rho_{pt}(r) d\Sigma = 2\pi\int_x^\infty r\rho_{pt}(r) dr$$

where r is the distance from an element of the surface $d\Sigma$ to the point of space considered.

Hence it follows that

$$\rho_{pt}(r) = -\frac{1}{2\pi r}\frac{d\rho_{pl}(r)}{dr}$$

Hence, for an isotropic point source

$$M_s^m(t,\sigma,r) = \frac{\sigma+1}{2\pi^{3/2}r}\sum_\alpha \frac{a_\alpha^s a_\alpha^m}{\eta_\alpha N_\alpha}\exp(\sigma t-(\sigma+1)r/\eta_\alpha)+$$

$$+\frac{(\sigma+1)^3}{2\pi^{3/2}r}\int_0^{\pm\infty}\frac{N_{11}b_1^s b_1^m + N_{22}b_2^s b_2^m - N_{12}(b_1^s b_2^m + b_2^s b_1^m)}{\eta N_0}\exp(\sigma t-(\sigma+1)r/\eta)d\eta \qquad (20)$$

In the case of an arbitrary time dependence of the source power it can be represented in the form of a Fourier integral

$$W(t) = \int_{-\infty}^{+\infty} W_\omega \exp(i\omega t)d\omega, \quad W_\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} W(t)\exp(-i\omega t)dt$$

and, by virtue of the linearity of the problem, we can consider the distribution of the macroscopic parameters of the gas as the superposition of corresponding quantities produced by the individual harmonics

$$M_s^m(t,r) = \int_{-\infty}^{+\infty} W_\omega M_s^m(t,i\omega,r)\exp(i\omega t)dt$$

We will analyse the solution obtained.

In Fig. 1 we show the regions C_1, C_2 and C_3 of variation of the parameter σ, in which the dispersion equation has two, four and six roots respectively. For the negative half-space, Imσ, the pattern has a form that is symmetrical about the real axis. A numerical analysis shows that, when σ approaches the boundary of these regions from the inside, the imaginary part of one of the pairs of roots tends to zero, and the solutions corresponding to it transfer into the solutions of the continuous spectrum, in which case the general solution, i.e. the sum of the solutions of the continuous and discrete spectra, remains a continuous function of σ.

In the immediate vicinity of the source ($r \ll |\sigma + 1|$), the solutions of the continuous spectrum make the main contribution to expression (20). The value of the integrals is then determined by the small values of η, for which

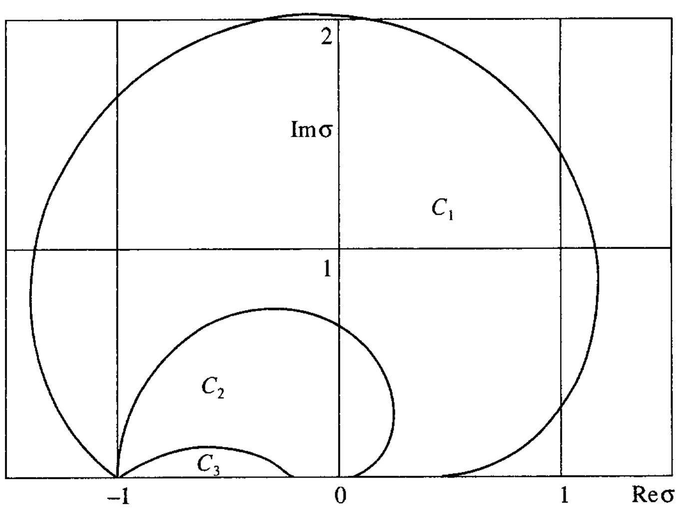

Fig. 1

$$\lambda_p = 0,\quad N_{11} = \frac{9}{8}(\sigma+1)^2,\quad N_{12} = \frac{5}{8}(\sigma+1)^2,\quad N_{22} = \frac{25}{8}(\sigma+1)^2$$

$$N_0 = \frac{25}{8}(\sigma+1)^4\eta,\quad \int_0^\infty \exp(-(\sigma+1)r/\eta)\frac{d\eta}{\eta^2} = \frac{1}{r(\sigma+1)}$$

Correspondingly

$$\Delta T_h = \frac{9}{50\pi^{3/2}r^2},\quad \Delta T_h^\omega = \Delta T_\omega = \frac{1}{5\pi^{3/2}r^2},\quad \Delta N_h = \Delta T_p = -\frac{1}{10\pi^{3/2}r^2}$$

$$\Delta T_\upsilon = \Delta T_\upsilon^\upsilon = \Delta T_h^\upsilon = \Delta T_\upsilon^\omega = \Delta T_\omega^\upsilon = -\Delta N_\upsilon = -\Delta T_p^\upsilon = \frac{1}{6\pi^{3/2}r^2}$$

Hence, the distribution of the majority of the moments mentioned in the region of the source is independent of σ and is determined solely by its instantaneous power. An exception is $\Delta T_p^\omega = \Delta N_\omega$, the values of which in the limit considered are specified by the following terms in the expansion in η, which make a contribution to the distribution of these moments proportional to $1/r$.

As one moves away from the source, the second term in expression (20) decays more rapidly than the first. Hence, in the limit as $r \to \infty$ the distribution of M_s^m is determined by the solutions of the discrete spectrum (if such exist).

We are particularly interested in analysing the behaviour of the solution in the case of small values of σ. Substituting into Eq. (9) the obvious asymptotic representation

$$\lambda_c(z) = -\sum_{n=0}^{\infty}\frac{(2n-1)!!}{2^n z^{2n}}$$

we obtain, apart from the first non-vanishing terms in z

$$\frac{7}{20\eta^4} - \frac{7\sigma}{10\eta^2} + \sigma^3 = 0$$

Hence we obtain

$$\eta_{\pm 1} = \pm\sqrt{7/10}\,\sigma^{-1},\quad \eta_{\pm 2} = \pm(2\sigma)^{-1/2}$$

Correspondingly

$$N_{\pm 1}\sqrt{\pi}\eta_{\pm 1} = \frac{49}{625\sigma},\quad N_{\pm 2}\sqrt{\pi}\eta_{\pm 2} = \frac{7}{50}$$

$$a_{\pm 1}^1 = \frac{2}{25},\quad a_{\pm 2}^1 = -a_{\pm 11}^4 = -a_{\pm 2}^4 = -\frac{1}{5}$$

$$a_{\pm 1}^2 = \frac{3}{25},\quad a_{\pm 2}^2 = -\frac{3}{10},\quad a_{\pm 1}^3 = \frac{1}{50},\quad a_{\pm 2}^3 = -\frac{1}{20}$$

Hence, in the limit as $\sigma \to 0$

$$\Delta T_h = \Delta T_h^{as} + I_{11}$$

$$\Delta T_h^\upsilon = \Delta T_\upsilon = \Delta T_h^{as} + \frac{5I_{11}+I_{12}}{6},\quad \Delta T_h^\omega = \Delta T_\omega = \Delta T_h^{as} + \frac{5I_{11}-I_{12}}{4}$$

$$\Delta T_\upsilon^\upsilon = \Delta T_h^{as} + \frac{25I_{11}+10I_{12}+I_{22}}{36},\quad \Delta T_\upsilon^\omega = \Delta T_\omega^\upsilon = \Delta T_h^{as} + \frac{25I_{11}-I_{22}}{24}$$

$$\Delta T_\omega^\omega = \Delta T_h^{as} + \frac{25I_{11}-10I_{12}+I_{22}}{16}$$

$$\Delta T_p = \Delta T_p^{as} - I_{12}, \quad \Delta T_p^{\upsilon} = \Delta T_p^{as} - \frac{5I_{12}+I_{22}}{6}, \quad \Delta T_p^{\omega} = \Delta T_p^{as} - \frac{5I_{12}-I_{22}}{4}$$

$$\Delta N_h = \Delta N_h^{as} - I_{12}$$

$$\Delta N_\upsilon = \Delta N_h^{as} - \frac{5I_{12}+I_{22}}{6}, \quad \Delta N_p = \Delta N_p^{as} + I_{22}, \quad \Delta N_\omega = \Delta N_h^{as} - \frac{5I_{12}-I_{22}}{4}$$

$$I_{ij} = \frac{1}{2\pi^{3/2}r}\int_0^\infty \frac{N_{ij}}{\eta N_0}\exp(-r/\eta)d\eta$$

which corresponds to the solution of the time-independent problem. In this case the functions

$$\Delta T_h^{as} = -\Delta N_h^{as} = 1/(7\pi r)$$

describe the distribution of the temperature and density of the gas molecules produced by a time-independent point source of heat at a fairly large distance from it, and is independent of the method by which the energy is excited; the functions

$$\Delta N_p^{as} = -\Delta T_p^{as} = 1/(7\pi r)$$

describe the asymptotic distribution of the temperature and density of the gas molecules produced by a time-independent particle source.

The results obtained can be used for a theoretical analysis of the features of heat and mass transfer in rarefied gases, in particular, when investigating the thermal effects of the interaction of a laser beam with a material, which is particularly important when investigating the phenomenon of thermal self-focusing and defocusing of a laser beam in an absorbing medium, particularly in the case when the characteristic heat liberation time is comparable with the time of the mean free path of the gas molecules.

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